A Continuous 3-D Medial Shape Model with Branching

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Outline

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• Review of Subdivision Surfaces
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Introduction

- Computing the medial axis is unstable
  - Lots of little branches, different topology for every shape
- Generating shapes from a fixed medial topology is stable
- Works great in 2-D
- In 3-D, have infinitely many singular points $\rightarrow$ infinitely many constraints, but finite number of parameters
The Generic 3-D Medial Axis

- Most points belong to flat sheets with exactly two spokes

(healthy left caudate)
The Generic 3-D Medial Axis

(a) edge curve

(b) branch curve

(c) fin point

(d) 6-junction
Catmull-Clark Subdivision

- Face points added with F-mask
- Edge points added with E-mask
- Old vertex positions updated with V-mask
Catmull-Clark Subdivision

- Subdivide once: quadrilaterals
- Subdivide twice: at most one extraordinary vertex
- Most patches are B-splines
- Evaluate rest with (Stam 1999)
Boundary Reconstruction

- Defined by a continuous medial axis \( m \) and radius function \( r \).
- Boundary recovered via:
  \[
  B = m + r \tilde{U}
  \]
  \[
  \tilde{U} = -\nabla r \pm \sqrt{1 - \|\nabla r\|^2} \hat{n}
  \]
- Here \( \nabla r \) is the Riemannian gradient:
  \[
  \nabla r = \begin{bmatrix} m_u & m_v \end{bmatrix} I_m^{-1} \begin{bmatrix} r_u \\ r_v \end{bmatrix}
  \]
Edge Curves

• Challenge: Edge boundary condition

$$\|\nabla r\| = 1$$

• Yushkevich's first solution (2003)
  ◆ Implicitly solve for a curve that satisfies it
  ◆ Different curve found for each object
    • Makes statistics on groups of objects hard

• Yushkevich’s second solution (2005)
  ◆ Parametrize the model with a potential field $\rho$ and solve a differential equation to recover $r$.
  ◆ Not enough free parameters for branches
Our Approach

• Start with a B-spline patch
Our Approach

- Interpolate in the $v$ direction
Our Approach

- Convert to an interpolating basis in $u$
Our Approach

• Replace the curve on the left with a control curve

\[ r_0(v) \quad r_1(v) \quad r_2(v) \quad r_3(v) \]
Our Approach

- Finish interpolating in the $u$ direction
Edge Curves

- Interpolation is given by the B-spline:
  \[(m, r) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} B P B^T \begin{bmatrix} 1 & v & v^2 & v^3 \end{bmatrix}^T\]

- Change to Catmull-Rom basis \(C\) on left:
  \[(m, r) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} C P' B^T \begin{bmatrix} 1 & v & v^2 & v^3 \end{bmatrix}^T, \quad P' = C^{-1} B P\]

- Now \(r(0, v)\) and \(r_v(0, v)\) are independent of \(P'_{0,j}\)

- Hold \(m_v(0, v)\), \(r_v(0, v)\) and \(m_u(0, v)\) fixed and solve for \(r_u(0, v)\)
Edge Curves

• Split $\nabla r$ into two components:

$$\nabla r^{(v)} = \frac{r_v}{\sqrt{G_m}} \quad \nabla r^{(\perp v)} = \frac{r_u G_m - r_v F_m}{\sqrt{G_m (E_m G_m - F_m^2)}}$$

• Solving $\|\nabla r\| = 1$ for $r_u$ yields:

$$r_u = \frac{1}{G_m} \left( r_v F_m \pm \sqrt{(G_m - r_v^2)(E_m G_m - F_m^2)} \right)$$

• Note $\nabla r^{(\perp v)}$ is always pointing inwards with the + solution
Edge Curves

- Replace $P'_{0,j}$ with control curve $r_0(v)$
- Interpolate remaining columns in $v$
  $$r_i(v) = \begin{bmatrix} P'_{i,0} & P'_{i,1} & P'_{i,2} & P'_{i,3} \end{bmatrix} B^T \begin{bmatrix} 1 & v & v^2 & v^3 \end{bmatrix}, \quad i = 1 \ldots 3$$
- Solve for $r_0(v)$
  $$r_0(v) = \frac{1}{C_{10}} \left( r_u(0,v) - C_{11} r_1(v) - C_{12} r_2(v) - C_{13} r_3(v) \right)$$
- Finish the interpolation
  $$r(u,v) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} C \begin{bmatrix} r_0(v) & r_1(v) & r_2(v) & r_3(v) \end{bmatrix}^T$$
Edge Curves

- Final Result
Edge Curves

- Interpolation is asymmetric in $u$ and $v$
- Therefore the mesh must have no corners
- Trivial example: regular grid with corners deleted
- Extraordinary vertices on the edge are also disallowed
  - All edge vertices will have valence 3, except at fin points
Edge Curves

- Complete model (healthy left ventricle)
Edge Curves

• Complete model (healthy left ventricle)
Branch Curves

- If we can get a succinct expression of the condition to enforce, can use exactly the same method
- Assume all 3 patches meet on the $u=0$ curve (share control points)
Branch Curves

- The tips of $-\nabla r$ and $U$ lie in the same plane in all three patches.
Branch Curves

• Project everything into this plane

• Need

\[ \theta^{(0)} + \theta^{(1)} + \theta^{(2)} = \pi \]

• And

\[ \phi^{(i)} = \theta^{(i)} + \theta^{(i \oplus 1)} \]
Branch Curves

- Using right triangles shows

\[ \nabla r^{(\perp v, i)} = -\sqrt{1 - \frac{r_v^2}{G_m}} \cos \theta^{(i)} \]

- Solving for \( r_u^{(i)} \) produces

\[ r_u^{(i)} = \frac{1}{G_m} \left( r_v F_m^{(i)} - \cos \theta^{(i)} \sqrt{(G_m - r_v^2) \left( E_m^{(i)} G_m - F_m^{(i)} \right)} \right) \]
Fin Points

- Edge curve meets branch curve
- Enforce edge condition on patch 0 as normal, take $\theta^{(0)}$ as fixed
- Use a control curve on $r$ to set $\theta^{(1)}$ and $\theta^{(2)}$ so they all sum to $\pi$
- Use another on $m$ to set $m_u^{(\perp v, 1)}$ and $m_u^{(\perp v, 2)}$ to bisect them
Fin Points

• 3 more constraints required at fin point itself (detailed in paper)

• But only 3! Can enforce by adjusting control points using the masks:

\[
\begin{bmatrix}
0 & \Delta r' & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -\Delta r' & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\] Adjusts \( r_u^{(0)}(0,0) \)

\[
\begin{bmatrix}
0 & \Delta m' & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -\Delta m' & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\] Adjusts \( m_u^{(0)}(0,0) \)

\[
\begin{bmatrix}
0 & -2\Delta r'' & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \Delta r'' & 0 & 0 \\
0 & -2\Delta r'' & 0 & 0
\end{bmatrix}
\] Adjusts \( r_{uu}^{(0)}(0,0) \)
Fin Points

• Final result
Conclusion

• Demonstrated a new, continuously defined, generative 3-D medial model
• First such model to support branches
• Restricted to a mesh without corners
  ♦ Bad for synthetic objects, good for biological
• Don’t explicitly enforce several additional inequality constraints needed to prevent overfolding, etc.
  ♦ Enforce during model fitting process
Questions?
Catmull-Clark Subdivision

• Handle extraordinary vertices via eigendecomposition (Stam 1999)

\[
P_1 = AP_0 \\
\bar{P}_1 = \bar{A}P_0 \\
\vdots \\
P_n = AP_{n-1} = A^n P_0 \\
\bar{P}_n = \bar{A}P_{n-1} = \bar{A}\bar{A}^{n-1} P_0
\]

• \( \varnothing + \circ = \bar{P}_1 \), \( \circ = P_1 \)
Catmull Clark Subdivision

- $\frac{3}{4}$ of domain is now a B-spline

\[
P_1 = A P_0
\]
\[
\bar{P}_1 = \bar{A} P_0
\]
\[
\vdots
\]
\[
P_n = A P_{n-1} = A^n P_0
\]
\[
\bar{P}_n = \bar{A} P_{n-1} = \bar{A} A^{n-1} P_0
\]
Catmull-Clark Subdivision

- No explicit subdivision required
- Diagonalize A:

\[ A = V \Lambda V^{-1} \]

\[ \bar{P}_n = \bar{A} A^{n-1} P_0 = \bar{A} V \Lambda^{n-1} V^{-1} P_0 \]

- \( V^{-1} P_0 \) can be computed once per patch
- \( \bar{A} V \) can be precomputed
Edge Curves

- Sympathetic overfolding sometimes appears on the right side
  - $C^2$ continuity does not help
- Choosing $P_{0,j} = \frac{1}{2}(P_{1,j} + P_{2,j})$, $\tilde{P}_{1,j} = \frac{11}{8} P_{1,j} - \frac{3}{8} P_{2,j}$ helps slow down $r$ without changing the edge
- Choosing $P_{0,j} = P_{2,j}$, $\tilde{P}_{1,j} = \frac{3}{2} P_{1,j} - \frac{1}{2} P_{2,j}$ gives infinite slowdown, but ill-defined tangent plane
Sympathetic Overfolding

- Using a 4\textsuperscript{th}-order spline lets us add an extra condition: a well-defined tangent plane via reparameterization